AdS/CFT boundary conditions and multi-trace perturbations for fermions.

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Abstract

Extending the results of a previous paper, we consider boundary conditions for spinor fields and other fields of non-zero spin in the AdS/CFT correspondence. We calculate the RG-flow induced by double trace perturbations dual to bulk spinor fields. For spinors there is a half-unit shift in the central charge in running from the UV to the IR, in accordance with the c-theorem.

To investigate the AdS/CFT correspondence [1] beyond leading order in the large-N expansion, we have to consider quantum loops in the bulk theory. In general this includes string loops, but to leading order in the α' expansion sub-leading order corrections are given by Supergravity loops. To perform such loop calculations requires a knowledge of what boundary conditions to impose on the bulk fields.

In a recent paper [8], we discussed the possible boundary conditions on scalar fields. In particular we calculated the one-loop Weyl anomaly for all possible boundary conditions. The asymptotic behaviour of the bulk scalar field has two components:

$$\phi = \alpha(x) z^{d-\Delta} + \beta(x) z^{\Delta} + \dots , \tag{1}$$

where Δ is the larger root of the equation $\Delta(\Delta - d) = m^2$. From a Hamiltonian point of view, after a change of variables to reproduce the usual inner-product on scalar fields, we found that diagonalising α and β correspond to Dirichlet and Neumann conditions, respectively. For Dirichlet conditions (and generic mixed boundary conditions) the central charge is as computed in [5, 6]. For Neumann conditions, however, there is a unit shift of in the central charge, relative to the central charge for Dirichlet conditions.

One situation in which the choice of boundary conditions is important is when we perform double-trace perturbations of the gauge theory that correspond to tachyonic fields in the Supergravity theory [4]. These break the conformal symmetry of the boundary gauge theory, and drive a renormalisation group flow. For tachyonic modes whose masses lie in an appropriate range, the difference between the ultraviolet and infrared fixed points corresponds to the choice of Dirichlet or Neumann boundary conditions for the bulk field [8].

In this letter we will extend these results to fermions and other fields of non-zero spin. For fermions we will show that running from the UV to the IR causes a shift of $\frac{1}{2}$ in the central charge (we will relate this to boundary conditions for the fermions). This shift in in accordance with the c-theorem [10, 9].

We will work in an AdS_{d+1} metric whose boundary is a d-dimensional Einstein metric $\hat{g}$ [6]. The metric is

$$ds^2 = G_{\mu\nu}\, dX^\mu\, dX^\nu = dr^2 + z^{-2}\, e^\rho \hat{g}_{ij}(x)\, dx^i dx^j \,, \quad e^{\rho/2} = 1 - C\, z^2 \,, \quad C = \frac{l^2 \hat{R}}{4\, d(d-1)} \,, \tag{2}$$

where $\hat{R}$ is the Ricci tensor on the boundary. The Euclidean action for a spin-1/2 fermion in this metric is

$$\int d^{d+1}x \sqrt{G} \bar{\psi} (\gamma^\mu D_\mu - m) \psi. \tag{3}$$

The spin-covariant derivative is defined via the funfbein

$$V_0^\alpha = \frac{1}{z} \delta_0^\alpha, \qquad V_i^\alpha = \frac{1}{z} e^{\rho/2} \tilde{V}_i^\alpha, \tag{4}$$

where $\tilde{V}_i^\alpha$ is the vierbein for the boundary metric. Making the change of variables $\psi = z^{d/2} e^{-d\rho/4} \tilde{\psi}$ causes the volume element in the path-integral to become the usual flat-

space one, and the kinetic term in the action acquires the usual form. The action can be written (discarding a non-essential boundary term)

$$\int d^{d+1}x\bar{\tilde{\psi}}\left(\gamma^0\partial_0 + ze^{-\rho/2}\gamma^i\tilde{D}_i - m\right)\tilde{\psi}. \tag{5}$$

The D_i derivative is spin-covariant with respect to the boundary metric. We impose the following boundary conditions on $\tilde{\psi}$:

$$Q_+\tilde{\psi}(0,x) = u(x) = Q_+u(x), \qquad \tilde{\psi}^\dagger(0,x)Q_- = u^\dagger(x) = u^\dagger(x)Q_-, \tag{6}$$

for some local projection operators $Q_\pm$. The remaining projections are represented by functional differentiation. The partition function takes the form

$$\Psi[u,u^\dagger] = \exp[f + u^\dagger\Lambda u], \tag{7}$$

and the Schrödinger equation that it satisfies can be written

$$\frac{\partial}{\partial r_0}\Psi = -\int d^dx\left(u^\dagger Q_- + \frac{\delta}{\delta u}Q_+\right)h\left(Q_+u + Q_-\frac{\delta}{\delta u^\dagger}\right)\Psi, \tag{8}$$

where $h = \tau e^{-\rho/2}\gamma^0\gamma^i\tilde{D}_i - \gamma^0 m$. If we make the specific choice $Q_\pm = \frac{1}{2}(1\pm\gamma^0)$, we can write (8) as

$$\frac{\partial}{\partial r_0}\Psi = -\left[mu^\dagger\frac{\delta}{\delta u^\dagger} - m\frac{\delta}{\delta u}u - \tau e^{-\rho/2}u^\dagger\gamma\cdot\tilde{D}u + \tau e^{-\rho/2}\frac{\delta}{\delta u}\gamma\cdot\tilde{D}\frac{\delta}{\delta u^\dagger}\right]\Psi. \tag{9}$$

Acting on (7) this implies that

$$\dot{\Lambda} = -2m\Lambda + \tau e^{-\rho/2}\gamma\cdot\tilde{D} - \Lambda^2\tau e^{-\rho/2}\gamma\cdot\tilde{D}, \qquad \dot{f} = \tfrac{1}{2}\mathrm{Tr}(-m + \Lambda\tau e^{-\rho/2}\gamma\cdot\tilde{D}). \tag{10}$$

Solving this in terms of Bessel functions we find that in momentum space(assuming $m \geq 0$)

$$\Lambda = \frac{I_{m+1/2}(p\tau)}{I_{m-1/2}(p\tau)}\mathcal{P}, \tag{11}$$

where $\frac{1}{2}(1\pm\mathcal{P})$ are projectors onto +ve/-ve eigenvalues of the operator $\gamma\cdot\tilde{D}$. As with scalar fields, to get the correct scaling dimension as $\tau\to 0$ requires discarding terms of order less than τ^{2m} in the asymptotic expansion of Λ[1]. After removing the unwanted terms, we have the asymptotic behaviour as $\tau \to 0$

$$\Lambda \sim \tau^{2m}p^{2m}\mathcal{P}, \tag{12}$$

and to get a finite wave-functional we must perform a wave-function renormalisation $\psi \to \tau^{-m}\psi$. The scaling dimension of the boundary field is then $d/2+m$ as required. We

[1] That this prescription is the correct one is more easily seen if we construct the wave-functional by integrating from $z=0$ to $z=\tau'$ with τ' a large regulator, as in [5], giving the correct scaling dimension as $\tau'\to\infty$ without any additional renormalisation. The wave-functional tends to a delta functional as $\tau'\to 0$, as it should. The renormalisation we use here was also discussed in [3].

could have achieved an identical result by diagonalising $\tau^{m-d/2}Q_+\psi$ in the first instance; our approach has the advantage of removing cutoff dependence from the functional inner-product. Note that the choice $Q_\pm = \frac{1}{2}(1 \pm \gamma^0)$ was forced on us, and to consider other boundary conditions requires a functional integration over boundary values. For example, if we add a boundary term $\int(u^\dagger v - v^\dagger u)$ and integrate over u and $u^\dagger$ this changes the boundary conditions to

$$Q_-\tilde{\psi}(0,x) = v(x) = Q_-v(x), \qquad \tilde{\psi}^\dagger(0,x)Q_+ = v^\dagger(x) = v^\dagger(x)Q_+. \tag{13}$$

The renormalised fields v and $v^\dagger$ represent the canonical conjugates of the fields we diagonalised before. They have scaling dimension $d/2 - m$ and unitarity in the bulk indicates that the boundary conditions (13) are normalisable for $m \leq \frac{1}{2}$. For $m > 1/2$ there is only one admissible boundary condition. Recall that this assumed $m \geq 0$.

A double-trace perturbation of the boundary theory corresponds to adding a quadratic boundary term to the action:

$$I_{CFT} \to I_{CFT} + f\int u^\dagger u. \tag{14}$$

For spin-1/2 operators of scaling dimension $d/2 + m$ where $|m| \leq \frac{1}{2}$ this drives a RG-flow from a UV fixed point at $f = 0$ to an IR fixed point at $f = \infty$. From the point of view of the holographically dual bulk fields the UV fixed point corresponds to the boundary conditions (6) and the IR fixed point corresponds to the boundary conditions (13). A double-trace perturbation of this kind was considered in [2].

To calculate the central charge associated with the Weyl anomaly we expand (11) in terms of the positive-definite operator $(\gamma \cdot \tilde{D})^2$:

$$\Gamma = \gamma \cdot \tilde{D} \sum_{n=0}^{\infty} d_n(r_0)(\gamma \cdot \tilde{D})^{2n}. \tag{15}$$

Notice that the coefficients d_n *all* vanish as $r_0 \to -\infty$. The equation (10) is easily solved in terms of Bessel functions, but to regulate the expression for the free energy f we again use a heat-kernel expansion

$$\mathrm{Tr}(-m + \Gamma\tau\gamma \cdot \tilde{D}) = \left(\sum_{n=0}^{\infty} d_n(r_0)\left(-\frac{\partial}{\partial s}\right)^{n+1} - m\right) \mathrm{Tr}\exp\left(-s(\gamma \cdot \tilde{D})^2\right), \tag{16}$$

where the heat-kernel has a Seeley-de Witt expansion

$$\mathrm{Tr}\ \exp\left(-s\left(-s(\gamma \cdot \tilde{D})^2\right)\right) = \int d^dx\,\sqrt{\hat{g}}\frac{1}{(4\pi s)^{d/2}}\left(a_0 + s\,a_1(x) + s^2\,a_2(x) + s^3\,a_3(x) + ..\right), \tag{17}$$

The contribution proportional to the a_2 coefficient is finite as $s \to 0$ and $r_0 \to -\infty$ and determines the anomaly, which is therefore proportional to m. But since $m = \Delta - 2$ we have as before

$$\delta\mathcal{A} = -\frac{\Delta - 2}{32\pi^2}\,a_2\,. \tag{18}$$

This calculation of the anomaly assumed boundary conditions that diagonalised $\tilde{\psi}^\dagger(0,x)\frac{1}{2}(1-\gamma^0)$ and $\frac{1}{2}(1+\gamma^0)\tilde{\psi}(0,x)$, but we can change to other boundary conditions by performing a functional integration over the boundary values. For example to diagonalise $\tilde{\psi}^\dagger(0,x)\frac{1}{2}(1+\gamma^0)$ and $\frac{1}{2}(1-\gamma^0)\tilde{\psi}(0,x)$ we integrate over u and $u^\dagger$. There is a correction to the free energy given by

$$e^{2\delta f} = \det(\tfrac{1}{2}(1+\gamma^0)\Lambda). \tag{19}$$

From this we calculate that

$$\frac{\partial}{\partial r_0}\delta f = \mathrm{Tr}\left(\tfrac{1}{2}(1+\gamma^0)\tfrac{-2m\Lambda+\tau e^{-\rho/2}\gamma\cdot\tilde{D}-\Lambda^2\tau e^{-\rho/2}\gamma\cdot\tilde{D}}{2\Lambda}\right), \tag{20}$$

and inserting the asymptotic behaviour of Λ as $\tau\to 0$ we find the correction

$$\frac{\partial}{\partial r_0}\delta f = \mathrm{Tr}(-1/2), \tag{21}$$

leading to the correction

$$\delta\mathcal{A} = -\frac{1/2}{32\pi^2}\,a_2\,. \tag{22}$$

We conclude that the central charge is decreased by 1/2 as we run from the UV to the IR. Here we have used the definition of the central charge given in [10], and the result is in accordance with the proposed c-theorem in even dimensions [10, 9].

We can also consider more general boundary conditions. If instead of (13) we impose the conditions

$$P_-\tilde{\psi}(0,x) = v(x) = P_-v(x), \qquad \tilde{\psi}^\dagger(0,x)P_+ = v^\dagger(x) = v^\dagger(x)P_+, \tag{23}$$

for some local projectors $P_\pm$, the correction to the free energy is given by

$$e^{2\delta f} = \det(\tfrac{1}{2}(1+\gamma^0)\Lambda + A^{-1}C), \tag{24}$$

with $A = \{P_-, Q_+\}$ and $C = [P_-, Q_+]$. This leads to the same non-zero correction to the central charge as before, if and only if A or C vanishes. Otherwise there is no correction to the result (18).

For spin-3/2 fermions the action can be written in the form (3) (with a vector index on the fermion fields) and the Schrödinger equation also takes the form (8). However, unitarity considerations rule out the boundary conditions (13) for any value of the mass (in fact they rule out $|m| < 1/2$ completely).

For a scalar field the corresponding calculation was performed in [8] and we found a unit correction to the central charge. Since bosonic fields of higher spin satisfy a Schrödinger equation that has exactly the same form, there is a correction of exactly the same amount, although unitarity severely restricts the scaling dimensions for which a non-trivial flow is possible. So, for example, in four dimensions there could in principle be a flow for bulk vectors with masses saturating the BF-bound (ie. $m^2 = -1$) though such fields do not appear in the spectra of any known supergravity compactifications.

It is interesting that choosing the "irregular" boundary conditions causes the Weyl anomaly to vanish for the scalar and spinor fields of the doubleton representation of

SU(2,2/4). These fields decouple from the spectrum and were not included in the calculation of [6]. However they correspond to the decoupled $U(1)$ factor that makes the boundary gauge group $SU(N)$ instead of $U(N)$, and their contributions to the anomaly should therefore be included if we think about interpolating between large N and $N = 1$. This also happens for the doubleton representation of OSp(8/4).

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